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Fluctuation Theory for a Three-Dimensional Model of Maxwellian Molecules

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Under suitable assumptions, a functional central limit theorem is obtained for a three-dimensional model of Maxwellian molecules. This model is related to a nonlinear Boltzmann-type equation. It will be proved that the family of the distributions induced by fluctuation processes converges weakly.

KEY WORDS: Boltzmann equation; fluctuation; nonlinear evolution equation; Markov process; weak convergence.

1. INTRODUCTION

The study of large systems of interacting particles leads to nonlinear evolution equations. Maxwell and Boltzmann considered a diluted environment consisting of a large number n of balls with radius $\frac{1}{\sqrt{n}}$. Thus, they obtained a $(\mathbf{R}^6)^n$-valued Markov process in which only the initial positions and velocities are random. Unfortunately, the infinitesimal generator cannot be expressed properly. To get around such a problem, one of the modifications one can make is to substitute infinitesimal points for small balls; but in this case we must consider collisions between those points. Bezandry *et al.*[(1)] studied the modified Maxwell case. Indeed, they proved a result to the convergence in law of empirical processes associated with general interacting kernels.

This work is concerned with the fluctuation problem for the following three-dimensional model (see, e.g., ref. 2) which is fairly close to a realistic

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model known as Maxwellian molecules in interaction associated with the nonlinear equation of type,

$$\frac{d}{dt}\langle \lambda_t^N, \phi\rangle = \langle \lambda_t^N, \Gamma^N\phi\rangle + \langle \lambda_t^N \otimes \lambda_t^N, \Lambda_1\phi\rangle, \qquad \text{with} \quad \lambda_0^N = \mu \tag{1}$$

where, $N > 0$, is a fixed parameter, λ_t^N and μ are probability measures on $\mathbf{R}^3 \times \mathbf{R}^3$, and the brackets $\langle ., . \rangle$ are used to represent the integrals. The operator Γ^N and Λ_1 are defined as follows,

$$\Gamma^N\phi(x, v) = N\left[\phi\left(x + \frac{v}{N}, v\right) - \phi(x, v)\right] \tag{2}$$

$$\Lambda_1\phi(x, v; x', v') = \frac{1}{8\pi}\int_{S^2} [\varepsilon^\sigma\phi(x, v; x', v'))]\, d\sigma \tag{3}$$

where,

$$\varepsilon^\sigma\phi(x, v; x', v') = \phi(x, \tilde{v}(\sigma)) - \phi(x, v) + \phi(x', \tilde{v}'(\sigma)) - \phi(x', v')$$

and

$$\tilde{v}(\sigma) = v - \sigma(v - v')\,\sigma, \qquad \tilde{v}'(\sigma) = v' + \sigma(v - v')\,\sigma$$

In the definition of Λ_1, $d\sigma$ denotes normalized surface measure on the unit sphere S^2. The function ϕ is taken smooth enough in such a way that Γ^N is well-defined and that the integrals make sense. It is important to note that Eq. (1) is an asymptotic version of the following nonlinear equation,

$$\frac{d}{dt}\langle \lambda_t, \phi\rangle = \langle \lambda_t, \Gamma\phi\rangle + \langle \lambda_t \otimes \lambda_t, \Lambda_1\phi\rangle \tag{4}$$

where, $\Gamma\phi(x, v) = v . \nabla_x\phi(x, v)$.

This work actually represents an intermediate step for the study of the fluctuation problem associated to Eq. (4). The spatially homogeneous case (i.e., position neglected) is well-known and has been studied by Uchiyama in ref. 3 and also Ferland *et al.*[4] To get the fluctuation convergence, we shall follow their approach. We shall first prove the tightness of the fluctuation processes, then that any limiting law solves any associated martingale problem, and finally a uniqueness of a solution of the martingale problem.

2. MODEL AND NOTATION

2.1. Notation

Consider a sequence $\{W^{n,N}(t), t \in [0,T]\}_{n \geqslant 2}$ of Markov processes associated to Eq. (1). The nth process takes its values in $(\mathbf{R}^3 \times \mathbf{R}^3)^n$. Indeed, we have, $W^{n,N}(t) = (W_1^{n,N},..., W_n^{n,N}(t))$ with, $W_j^{n,N}(t) = (X_j^{n,N}(t), V_j^n(t))$. The vector $X_j^{n,N}(t)$ is the position component of $W_j^{n,N}(t)$ and $V_j^n(t)$ is its velocity component. The nth process is governed by the following generator:

$$G_n^N f(w_1,..., w_n) = L_n^N f(w_1,..., w_n) + H_n f(w_1,..., w_n) \tag{5}$$

with

$$L_n^N f(w_1,..., w_n) = N \sum_{j=1}^{n} \Gamma^{N_j} f(w_1,..., w_n),$$

$$H_n f(w_1,..., w_n) = \frac{1}{n} \sum_{i \neq j} \Lambda^{ij} f(w_1,..., w_n)$$

The operators Γ^{N_j} and Λ^{ij} are analogous to Γ^N and Λ_1, that is,

$$\Gamma^{N_j} f(w_1,..., w_n) = f\left(w_1,..., \left(x_j + \frac{v_j}{N}, v_j\right),..., w_n\right) - f(w_1,..., w_n)$$

$$\Lambda^{ij} f(w_1,..., w_n) = \frac{1}{4\pi} \int_{S^2} [f^{i,j,\sigma}(w_1,..., w_n) - f(w_1,..., w_n)]\, d\sigma$$

The function $f^{i,j,\sigma}$ is obtained from f by replacing the variables $w_i = (x_i, v_i)$ and $w_j = (x_j, v_j)$ by $\tilde{w}_i = (x, \tilde{v}(\sigma))$ and $\tilde{w}_j = (x_j, \tilde{v}'(\sigma))$ respectively.

2.2. Model

Let us describe the model studied in this work. If we denote by $w^{n,N} = (w_1^{n,N},..., w_n^{n,N})$, with $w_i^{n,N} = (x_i^{n,N}, v_i)$, the initial state vector, then the evolution scheme is the following:

(a) The vector $w^{n,N}$ remains unchanged for a random time τ which is exponentially distributed with parameter $Nn + (n-1)$.

(b) At the end of this random time, a jump may occur. It occurs in the component positions with probability $\frac{nN}{nN+(n-1)}$. In this case, the change is the following: a component j is chosen uniformly among the n possibilities and the vector $x_j^{n,N}$ is replaced by $x_j^{n,N}+\frac{v_j}{N}$. If the jump occurs in the component velocities, a couple (i, j) is chosen uniformly and velocities (v_i, v_j) are replaced by a random vector $(\tilde{v}_i, \tilde{v}_j)$ whose distribution is given by a transition probability defined by

$$Q(v_i, v_j : B) = \frac{1}{4\pi} \int_{S^2} \mathbf{I}_B(\tilde{v}_i(\sigma), \tilde{v}_j(\sigma))\, d\sigma \tag{6}$$

where $\mathbf{I}_B$ is the characteristic function of $B \in \mathscr{B}(\mathbf{R}^6)$ and $\tilde{v}_i$, $\tilde{v}_j$, and $d\sigma$ are as stated in Section 1.

(c) The previous steps repeat indefinitely (with the new vector and independent exponential times).

We consider a system of Markov processes $\{W^{n,N}(t), t \in [0, T]\}$ and suppose that they are all defined on a common probability space $(\boldsymbol{\Omega}, \mathscr{A}, \mathbf{P})$ and that their paths are right continuous with left-hand limit (cadlag). The empirical distribution at time t: $\mu_t^{n,N} = \frac{1}{n} \sum_{j=1}^n \delta_{W_j^{n,N}(t)}$ of $W^{n,N}(t)$ and $\{\mu_t^{n,N}, t \in [0, T]\}$ is called the empirical process associated with $\{W^{n,N}(t), t \in [0, T]\}$. In the previous work,[1] we established the following theorem for general kernels.

Theorem 2.2.1. Let μ a probability measure on $\mathbf{R}^3 \times \mathbf{R}^3$ and assume that,

(a) There exists a contant $L > 0$ such that,

$$\int_{\mathbf{R}^3 \times \mathbf{R}^3} \|\tilde{v} - v\|\, Q(v, v', d\tilde{v}, d\tilde{v'}) \leqslant L(1 + \|v\| + \|v'\|)$$

for any v and v' in $\mathbf{R}^3$.

(b) $\sup_n \mathbf{E}[\frac{1}{n} \sum_{j=1}^n \|W_j^{n,N}(0)\|] < \infty$.

(c) $\mu_0^{n,N}$ converges in distribution to μ.

Then the empirical processes $\{\mu_t^{n,N}, \in [0, T]\}$ converge in distribution to a deterministic process $\{\lambda_t^N, t \in [0, T]\}$ which is the unique solution of (1). The scaled fluctuation of $\mu_t^{n,N}$ about λ_t^N is given by,

$$\eta_t^{n,N} = \sqrt{n}\, [\mu_t^{n,N} - \lambda_t^N]$$

The process $\{\eta_t^{n,N}, t \geqslant 0\}$ is a measure-valued temporally inhomogeneous Markov process. We name it the (n-particles) fluctuation process.

3. FLUCTUATIONS PROCESSES

A priori, the fluctuation $\eta_t^{n,N}$ takes values in the space of signed measures, but the limiting process(if it exists) is no longer signed-measure; it is an $H^{-4}(\mathbf{R}^6)$-process as shown in the following lemma:

Lemma 3.1. Under hypotheses (a) and (b) of Theorem 2.2.1 together with the notation above, we have:

(a) $\forall t \geqslant 0$ and $\langle \lambda_t^N, \|x\| \rangle < \infty$

(b) $\forall n \geqslant 2$, $\eta^{n,N}$ is a process taking its values in the Sobolev space $H^{-4}(\mathbf{R}^6)$.

Let us introduce Sobolev spaces, which are distribution spaces and Hilbert spaces. They play an important role in partial differential equations and in variational calculus. As shown in Lemma 3.1, the Sobolev space $H^4(\mathbf{R}^6)$ (or its dual $H^{-4}(\mathbf{R}^6)$) will be used in this paper, because it is the most appropriated to the present situation.

The Sobolev space of order m is defined as, $H^m(\mathbf{R}^n) = \{u \in L^2(\mathbf{R}^n) : (\forall \alpha \in N^n, |\alpha| \leqslant m)\ \partial^\alpha u \in L^2(\mathbf{R}^n)\}$. It is well-known the Sobolev space $H^m(\mathbf{R}^n)$, is a Hilbert space equipped with the following inner product, $\langle u, v \rangle_m = \sum_{|\alpha| \leqslant m} \int_{\mathbf{R}^n} \partial^\alpha u\, \overline{\partial^\alpha v}\, dx$. Now let us prove Lemma 3.1 assuming $m = 4$ and $n = 6$ and then considering the Hilbert space $H^4(\mathbf{R}^6)$. We shall denote its norm as $\|.\|_4$(respectively $\|.\|_{-4}$ the norm of $H^{-4}(\mathbf{R}^6)$).

Proof. Part (a) of this lemma is straightforward. It follows immediately from Lemma 2.2 (see, e.g., ref. 1). To establish part (b), we fix ω, n, N, and t; and consider a function $\phi \in H^4(\mathbf{R}^6)$. Then, we have to evaluate the quantity $|\langle \eta_t^{n,N}(\omega), \phi \rangle|$. We have the following,

$$|\langle \eta_t^{n,N}, \phi \rangle| \leqslant \sqrt{n} \left[\int_{\mathbf{R}^6} |\phi(x) - \phi(0)|\, \mu_t^{n,N}(dx) + \int_{\mathbf{R}^6} |\phi(x) - \phi(0)|\, \lambda_t^N(dx) \right]$$

Let us approximate $|\phi(x) - \phi(0)|$. By ref. 5, Lemma 6.95, p. 216, we have, $|\phi(x) - \phi(0)| \leqslant C\, \|\phi\|_4\, \|x\|^\alpha$. Assume $\alpha = \frac{1}{2}$, hence,

$$|\langle \eta_t^{n,N}, \phi \rangle| \leqslant C\sqrt{n} \left[\int_{\mathbf{R}^6} \sqrt{\|x\|}\, \mu_t^{n,N}(dx) + \int_{\mathbf{R}^6} \sqrt{\|x\|}\, \lambda_t^N(dx) \right] \|\phi\|_4$$

$$|\langle \eta_t^{n,N}, \phi \rangle| \leqslant C\sqrt{n} \left[\frac{1}{n} \left(\sum_{j=1}^{n} \|W_j^{n,N}(t)\| \right)^{\frac{1}{2}} + (\langle \lambda_t^N, \|x\| \rangle)^{\frac{1}{2}} \right] \|\phi\|_4$$

Since $\langle \lambda_t^N, \|x\| \rangle$ is finite by part (a) of the lemma, we conclude that, $\eta_t^{n,N}(\omega) \in H^{-4}(\mathbf{R}^6)$. Moreover, $\langle \eta_t^{n,N}, \phi \rangle$ is a real random variable for all

$\phi \in H^4(\mathbf{R}^6)$. Since $H^4(\mathbf{R}^6)$ generates the Borel σ-field of $H^{-4}(\mathbf{R}^6)$, this is enough to get the conclusion.

The following two lemmas play an important role throughout this paper. For any given probability measure μ on $\mathbf{R}^6$, we define an integral operator on $H^4(R^6)$ as follows: $\mathscr{L}(\mu)$: $H^4(\mathbf{R}^6) \to H^4(\mathbf{R}^6)$, such that,

$$\mathscr{L}(\mu)\,\phi(x,v) = \int_{\mathbf{R}^6} \mu(dx',dv') \int_{S^2} \varepsilon^\sigma \phi(x,v;x',v') \frac{d\sigma}{4\pi}$$

where ε^σ is given by Eq. (3).

Lemma A. The operator defined above, $\mathscr{L}(\mu)$ is continuous, that is, there exists $C > 0$ such that, $\|\mathscr{L}(\mu)\,\phi\|_4 \leqslant C\,\|\phi\|_4$, $\forall \phi \in H^4(\mathbf{R}^6)$.

Proof. It is not hard to see that when $\phi \in H^4(\mathbf{R}^6)$. The function $\mathscr{L}(\mu)\,\phi$ belongs to $H^4(\mathbf{R}^6)$. Indeed this fact is given by the fact that μ is a probability measure. Now, we have, $\|\mathscr{L}(\mu)\,\phi\|_4 \leqslant \int_{\mathbf{R}^6}\int_{S^2} \|\varepsilon^\sigma\phi(.,.;x',v')\|_4^2 \frac{d\sigma}{4\pi}\,\mu(dx',dv')$. To complete the proof, we just have to show that, there exists a constant $C > 0$, independent of x and v such that, $\|\varepsilon^\sigma\phi(.,.;x',v')\|_4^2 \leqslant C\,\|\phi\|_4^2$. Indeed, to find $\|\varepsilon^\sigma\phi(.,.;x',v')\|_4^2$, we have to find all derivatives of order less than or equal 4 of $\varepsilon^\sigma\phi$ considered as a function of only x and v. $\nabla_{(x,v)}\varepsilon^\sigma\phi(x,v;x',v') = (\nabla_x\varepsilon^\sigma\phi(x,v;x',v'), \nabla_v\varepsilon^\sigma\phi(x,v;x',v'))$. Such an operator contains all first derivatives of ϕ. Therefore using Cauchy–Schwarz inequality and considering the square on both side, it turns out that, there exists a constant $C_1 > 0$, such that, $\|\nabla\varepsilon^\sigma\phi\|^2 \leqslant C_1\,\|\nabla\phi\|^2$. The second derivatives of $\varepsilon^\sigma\phi$ are given by the matrix operators,

$$D^2_{(x,v)}\varepsilon^\sigma\phi(x,v;x',v') = \begin{pmatrix} D^2_{xx}\varepsilon^\sigma\phi & D^2_{xv}\varepsilon^\sigma\phi \\ D^2_{vx}\varepsilon^\sigma\phi & D^2_{vv}\varepsilon^\sigma\phi \end{pmatrix}.$$

The derivative operators: $D^2_{xx}\varepsilon^\sigma\phi, D^2_{xv}\varepsilon^\sigma\phi, D^2_{vx}\varepsilon^\sigma\phi, D^2_{vv}\varepsilon^\sigma\phi$ contain all derivatives of ϕ. Using again Cauchy–Schwarz inequality and considering the square on both side, it follows that there exists a constant $C_2 > 0$, such that, $\|D^2_{(x,v)}\varepsilon_\sigma\phi\|^2 \leqslant C_2\,\|D^2_{(x,v)}\phi\|^2$.

Using a similar reasoning, it can be shown that there exists $C > 0$ such that:

$$\sum_{|\alpha|\leqslant 4} \|D^\alpha_{(x,v)}\varepsilon^\sigma\phi\|^2 \leqslant C \sum_{|\alpha|\leqslant 4} \|D^\alpha_{(x,v)}\phi\|^2 \tag{7}$$

thus,

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \sum_{|\alpha|\leqslant 4} \|D^\alpha_{(x,v)}\varepsilon^\sigma\phi\|^2\,dx\,dv \leqslant C \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \sum_{|\alpha|\leqslant 4} \|D^\alpha_{(x,v)}\phi\|^2\,dx\,dv \tag{8}$$

which means that,

$$\|\varepsilon^{\sigma}\phi(.,.;x',v')\|_4 \leqslant C\,\|\phi\|_4. \tag{9}$$

Lemma B. Let $\phi \in H^4(\mathbf{R}^6)$ and consider the operator $\Lambda_1\phi$ given as in Eq. (3). Let us consider partial mappings given as,

$$\Gamma_1(\phi)\colon (x, v) \to \Lambda_1\phi(x, v; x', v') \qquad \text{and} \qquad \Gamma_2(\phi)\colon (x', v') \to \Lambda_1\phi(x, v; x', v')$$

Then the followings hold,

1. For x', v' fixed, $\Gamma_1(\phi)(x, v) \in H^4(\mathbf{R}^6)$,
2. For x, v fixed, $\Gamma_2(\phi)(x', v') \in H^4(R^6)$,
3. $\Lambda\phi \in H^4(\mathbf{R}^6) \otimes H^4(\mathbf{R}^6)$.

Proof. In this proof, we use relations Eq. (8) and Eq. (9) defined in the proof of Lemma A and the following formula,

$$\int_{S^{n-1}} f(\sigma)\,d\sigma = \int_{\mathbf{R}^n} I_\Delta f\left(\frac{x}{\|x\|}\right) dx \tag{10}$$

where $\Delta = \{x \in \mathbf{R}^n : 0 < \|x\| \leqslant 1\}$. Now let us consider such a formula for $n = 3$. Since (x, v) and (x', v') play symmetric roles, we have just to show (1) or (2). Let us prove (1) for instance. By using (10) it follows that,

$$\Gamma_1(\phi)(x, v) = \frac{1}{8\pi}\int_\Delta \varepsilon^{\frac{y}{\|y\|}}(x, v; x', v')\,dx\,dv$$

Using (9), we get, $\|\Gamma_1(\phi)\|_4 \leqslant \frac{C}{8\pi}\int_\Delta \|\phi\|_4\,dy$. Thus,

$$\|\Gamma_1(\phi)\|_4 \leqslant \frac{C}{8\pi}\,Mes(\Delta)\,\|\phi\|_{4(R^6)}$$

The assertion (3) is given by the fact that the following mapping is an isomorphim, $H^4(\mathbf{R}^6) \otimes H^4(\mathbf{R}^6) \to H^4(\mathbf{R}^{12})$, $f \otimes g \to (f, g)$, and the fact that $H^4(\mathbf{R}^{12}) \equiv H^4(\mathbf{R}^6) \times H^4(\mathbf{R}^6)$.

In the rest of this section we will derive a form of the infinitesimal generator of $\eta^{n,N}$ and its formal limit as $n \to +\infty$. For $\phi \in H^4(\mathbf{R}^6)$, $g \in C_0^\infty(\mathbf{R})$ (a C^∞-function vanishing off a compact set) and η a signed measure of the form:

$$\eta \equiv \sqrt{n}\,(\alpha - \lambda_t),\ \alpha = \frac{1}{n}\sum_{k=1}^{n} \delta_{w_k}, \qquad \text{where} \quad (w_1, w_2,\ldots, w_n) \in \mathbf{R}^{6n} \tag{11}$$

we set,

$$C_t^{n,N}(\eta, \phi, g) = \lim_{h \to 0} \frac{1}{h} [\mathbf{E}[g(\langle \eta_{t+h}^{n,N}, \phi \rangle) \mid \eta_t^{n,N} = \eta] - g(\langle \eta, \phi \rangle)] \tag{12}$$

(This is regarded as the infinitesimal generator for Markov process $\eta^{n,N}$ operating on the function, $\eta \to g(\langle \eta, \phi \rangle)$). If $\eta^{i,j,\omega}$ denote a signed measure obtained from η by replacing w_i, w_j by $\tilde{w}(\sigma), \tilde{w}'(\sigma)$ in Eq. (11), then $C_t^{n,N}(\eta; \phi, g)$ is expressed as:

$$C_t^{n,N}(\eta; \phi; g) = \frac{1}{4\pi n} \sum_{i<j} \int_{S^2} [g(\langle \eta^{i,j,\sigma,N}, \phi \rangle) - g(\langle \eta, \phi \rangle)] \, d\sigma$$
$$- \sqrt{n} \frac{d}{dt} \langle \lambda_t^N, \phi \rangle \, g'(\langle \eta, \phi \rangle)$$

Then, by observing, $n\langle \alpha \dot{\otimes} \alpha, \Lambda_1 \phi \rangle = \langle (\eta + \sqrt{n}\, \lambda_t)^{2\otimes}, \Lambda_1 \phi \rangle$; we can easily deduce that,

$$C_t^n(\eta, \phi, g) = \mathscr{A}_t^{n,N}(\phi)\, g'(\langle \eta, \phi \rangle) + \tfrac{1}{2} Q_t^{n,N}(\phi)\, g''(\langle \eta, \phi \rangle) + R_n \tag{13}$$

where

$$\mathscr{A}_t^{n,N}(\phi) = \langle \eta, \Gamma_1^N \phi \rangle + \frac{1}{\sqrt{n}} \langle \eta \otimes \eta, \Lambda_1 \phi \rangle + \langle \eta \otimes \lambda_t^N, \Lambda_1 \phi \rangle \tag{14}$$

$$Q_t^{n,N}(\phi) = \frac{1}{2} \langle \alpha, \Gamma_2^N \phi \rangle + \frac{1}{2} \langle \alpha \otimes \alpha, \Lambda_2 \phi \rangle \tag{15}$$

with

$$|R_n| \leqslant \frac{\|g'''\|_\infty}{12\sqrt{n}} (|\langle \alpha, \Gamma_3^N \phi \rangle| + |\langle \alpha \dot{\otimes} \alpha, |\Lambda_3 \phi| \rangle|)$$

In the above formulas, we have:

$$\alpha \dot{\otimes} \alpha = \frac{1}{n^2} \sum_{i \neq j} \delta_{w_i} \otimes \delta_{w_j}, \quad \Gamma_k^N \phi(x, v) = N \left[\phi\left(x + \frac{1}{N} v, v \right) - \phi(x, v) \right]^k$$

$$\Lambda_k \phi(x, v; x', v') = \frac{1}{8\pi} \int_{S^2} [\phi(x, \tilde{v}(\sigma)) - \phi(x, v) + \phi(x', \tilde{v}'(\sigma)) - \phi(x', v')]^k \, d\sigma$$

If we set $Q_t^N(\phi) = \frac{1}{2} (\langle \lambda_t^N, \Gamma_2^N \phi \rangle + \langle \lambda_t^N \otimes \lambda_t^N, \Lambda_2 \phi \rangle)$, and let $n \to \infty$ in Eq. (13), we get a formal limit of $C_t^{n,N}$: $(\langle \eta, \Gamma_1^N \phi \rangle + \langle \eta \otimes \lambda_t^N, \Lambda_1 \phi \rangle)\, g'(\langle \eta, \phi \rangle) + \frac{1}{2} Q_t^N(\phi)\, g''(\langle \eta, \phi \rangle)$, which should regulate the limiting process.

4. PRELIMINARY RESULTS

Let us introduce two additional conditions for $\{W^{n,N}(0), n \geqslant 2\}$:

$$(H_0) \quad \sup_n \mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n} \|W_j^{n,N}(0)\|^2\right] < \infty \tag{16}$$

$$(H_1) \quad \sup_n \mathbf{E}[\|\eta_0^{n,N}\|_{-4}^2] < \infty \tag{17}$$

The next proposition, which plays a crucial role in the whole story of this paper, asserts that the finiteness of (H_1) propagates.

Proposition 4.1. If H_0 and (H_1) hold, there exists a non-decreasing function K_t such that for all $t \geqslant 0$:

$$\sup_n \mathbf{E}[\|\eta_t^{n,N}\|_{-4}^2] \leqslant K_t$$

In order to prove this proposition, we need to introduce some ancillary quantities. Define for any $\phi \in H^4(R^6)$,

$$M_t^{n,N} \equiv M_t^{n,N}(\phi) = \langle \eta_t^{n,N}, \phi \rangle - \int_0^t \mathscr{A}_s^{n,N}(\phi)\, ds \tag{18}$$

$$S_t^{n,N} = S_t^{n,N}(\phi) = M_t^{n,N}(\phi)^2 - \int_0^t Q_s^{n,N}(\phi)\, ds \tag{19}$$

where $\mathscr{A}_s^{n,N}(\phi)$ and $Q_s^{n,N}(\phi)$ are defined by Eq. (14) and Eq. (15) respectively. Then $M_t^{n,N}$ and $S_t^{n,N}$ are martingales for the filtration $\mathscr{F}_t^{n,N} = \sigma(\mu_t^{n,N} : 0 \leqslant s \leqslant t)$. We know that the space $H^4(\mathbf{R}^6)$ is separable and that a complete orthogonal set $(\phi_k)_{k \geqslant 1}$ can be found in $H^4(\mathbf{R}^6)$, its Hilbert basis for instance. We will use this basis to express the dual norm of the fluctuation. For any ϕ_k, we have:

$$\begin{aligned}
&\mathbf{E}[(\langle \eta_t^{n,N}, \phi_k \rangle)^2] \\
&\quad \leqslant 2\mathbf{E}[M_t^{n,N}(\phi_k)^2] + 2\mathbf{E}\left[\left(\int_0^t \mathscr{A}_t^{n,N}(\phi_k)\, ds\right)^2\right] \\
&\quad = 2\mathbf{E}[S_t^{n,N}(\phi_k)] + 2\mathbf{E}\left[\int_0^t Q_s^{n,N}(\phi_k)\, ds\right] + 2\mathbf{E}\left[\left(\int_0^t \mathscr{A}_s^{n,N}(\phi_k)\, ds\right)^2\right] \\
&\quad \leqslant 2\mathbf{E}[S_t^{n,N}(\phi_k)] + 2\int_0^t \mathbf{E}[Q_s^{n,N}(\phi_k)]\, ds + 2t\int_0^t \mathbf{E}[\mathscr{A}_t^{n,N}(\phi_k)^2]\, ds
\end{aligned}$$

Since $(S_t^{n,N}(\phi_k))_{k\geqslant 1}$ is a martingale, we also have,

$$\mathbf{E}[S_t^{n,N}(\phi_k)] = \mathbf{E}[S_0^{n,N}(\phi_k)] = \mathbf{E}[|\langle \eta_0^{n,N}, \phi_k\rangle|^2]$$

Considering the sum over k, it follows that $\mathbf{E}[\|\eta_t^{n,N}\|_{-4}^2]$ is bounded by:

$$2\mathbf{E}[\|\eta_0^{n,N}\|_{-4}^2] + 2\int_0^t \sum_{k\geqslant 1} E[Q_s^{n,N}(\phi_k)]\, ds + 2t\int_0^t \sum_{k\geqslant 1} \mathbf{E}[\mathscr{A}_s^{n,N}(\phi_k)^2]\, ds$$

Upper-bounds for the two terms are given in the following lemma:

Lemma 4.1. For all $t\geqslant 0$, we have the following inequalities,

(a) $\sum_{k\geqslant 1} \mathbf{E}[Q_t^{n,N}(\phi_k)] \leqslant C\,\mathbf{E}[\frac{1}{n}\sum_{j=1}^n \|V_j^n(0)\|^2]^{\frac{1}{2}}$
(b) $\sum_{k\geqslant 1} \mathbf{E}[\mathscr{A}_t^{n,N}(\phi_k)^2] \leqslant C\,\mathbf{E}[\|\eta_t^{n,N}\|_{-4}^2]$.

Proof. (a) When x, v, x', v', and σ are fixed, using ref. 5, Lemma 6.95, p. 216, we can show that $\varepsilon^\sigma\phi(x, v; x', v')$ is a continuous linear functional on $H^4(\mathbf{R}^6)$ and the square of its norm in $H^{-4}(\mathbf{R}^6)$ is bounded by $C(\|v\| + \|v'\|)$. Parseval's identity gives,

$$\sum_{k\geqslant 1} |\varepsilon^\sigma\phi_k(x, v, x', v')|^2 \leqslant C(\|v\| + \|v'\|)$$

and therefore the following inequality holds:

$$\sum_{k\geqslant 1} \Lambda_2\phi_k(x, v; x', v') = \frac{1}{4\pi}\int_{S^2} \sum_{k\geqslant 1} |\varepsilon^\sigma\phi_k(x, v; x', v')|^2\, d\sigma \leqslant C(\|v\| + \|v'\|)$$

Similarly, we can show that: $\sum_{k\geqslant 1} \Gamma_2^N\phi_k(x, v) \leqslant C\,\|v\|$. On the other hand,

$$Q_t^{n,N}(\phi) = \tfrac{1}{2}\,[\langle \mu_t^{n,N}, \Gamma_2^N\phi\rangle + \langle \mu_t^{n,N}\otimes \mu_t^{n,N}, \Lambda_2\phi\rangle]$$

Then we can write:

$$\sum_{k\geqslant 1} \mathbf{E}[Q_t^{n,N}(\phi_k)]$$

$$\leqslant \frac{1}{2}\left(\mathbf{E}\left[\langle \mu_t^{n,N}\otimes \mu_t^{n.N}, \sum_{k\geqslant 1}\Lambda_2\phi_k\rangle\right] + \mathbf{E}\left[\left\langle \mu_t^{n,N}, \sum_{k\geqslant 1}\Gamma_2\phi\right\rangle\right]\right)$$

$$\leqslant \frac{1}{2n^2}\sum_{i,j=1}^n \mathbf{E}\left[\sum_{k\geqslant 1}\Lambda_2\phi_k(X_i^{n,N}(t), V_i^n(t), X_j^{n,N}(t), V_j^n(t))\right]$$

$$+\frac{1}{2n}\sum_{i=1}^n \mathbf{E}\left[\sum_{k\geqslant 1}\Gamma_2^N\phi(X_i^{n,N}(t), V_i^n(t))\right]$$

$$\leqslant \frac{C}{n^2} \sum_{i,j=1}^{n} E[\|V_i^n(t)\| + \|V_j^n(t)\|] + \frac{C}{n} \sum_{i=1}^{n} \mathbf{E}[\|V_i^n(t)\|]$$

$$\leqslant C\mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n} \|V_j^n(t)\|\right]$$

$$\leqslant C\mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n} \|V_j^n(t)\|^2\right]^{\frac{1}{2}} = C\mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n} \|V_j^n(0)\|^2\right]^{\frac{1}{2}}$$

The last equality follows from preservation of energy.

(b) From the definition of $\mathscr{A}_t^{n,N}(\phi)$, it follows that:

$$\mathscr{A}_t^{n,N}(\phi)^2 \leqslant C(|\langle \eta_t^{n,N} \otimes \lambda_t^N, \Lambda_1 \phi\rangle|^2 + \frac{1}{n} |\langle \eta_t^{n,N} \otimes \eta_t^{n,N}, \Lambda_1 \phi\rangle|^2$$

$$+ |\langle \eta_t^{n,N}, \Gamma_1^N \phi\rangle|^2)$$

We then have to bound each term properly, that is:

$$|\langle \eta_t^{n,N} \otimes \lambda_t^N, \Lambda_1 \phi\rangle| = |\langle \eta_t^{n,N}, \mathscr{L}(\lambda_t^N)\, \phi\rangle|$$

$$\leqslant \|\eta_t^{n,N}\|_{H^{-4}}\, \|\mathscr{L}(\lambda_t^N)\, \phi\|_4 \leqslant C\, \|\eta_t^{n,N}\|_{-4}\, \|\phi\|_4$$

Consequently,

$$\sum_{k\geqslant 1} |\langle \eta_t^{n,N} \otimes \lambda_t^N, \Lambda_1 \phi_k\rangle|^2 \leqslant C\, \|\eta_t^{n,N}\|_{-4}^2$$

Since the series is just the square of the norm in $H^{-4}(\mathbf{R}^6)$ of the random functional. By the same token, $\sum_{k\geqslant 1} \langle \eta_t^{n,N} \otimes \eta_t^{n,N}, \Lambda_1 \phi_k\rangle^2 \leqslant Cn\, \|\eta_t^{n,N}\|_{-4}^2$ and $\sum_{k\geqslant 1} \langle \eta_t^{n,N}, \Gamma_1^N \phi_k\rangle^2 \leqslant C\, \|\eta_t^{n,N}\|_{-4}^2$. Combining these inequalities, we get:

$$\sum_{k\geqslant 1} \mathbf{E}[\mathscr{A}_t^{n,N}(\phi_k)^2] \leqslant C\mathbf{E}[\|\eta_t^{n,N}\|_{-4}^2]$$

We now prove that the finiteness of (H_1) propagates. Set $y^{n,N}(t) = \mathbf{E}[\|\eta_t^{n,N}\|_{-4}^2]$. Using Lemma 4.1 , it follows that:

$$y^{n,N}(t) \leqslant 2y^{n,N}(0) + Ct\mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n} \|V_j^n(0)\|^2\right]^{\frac{1}{2}} + Ct \int_0^t y^{n,N}(s)\, ds$$

By Gronwall's Lemma we have,

$$y^{n,N}(t) \leqslant \left(2y^{n,N}(0) + Ct\mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n} \|V_j^n(0)\|^2\right]^{\frac{1}{2}}\right)\exp(Ct^2)$$

which proves the Proposition.

Proposition 4.2. The following holds,

$$\sup_n \mathbf{E}[\sup_{0\leqslant t\leqslant T} \|\eta_t^{n.N}\|_{-4}^2] < \infty$$

Proof. A martingale inequality gives,

$$\mathbf{E}[\sup_{0\leqslant t\leqslant T} M_t^{n,N}(\phi)^2] \leqslant 4\mathbf{E}[M_T^{n,N}(\phi)^2]$$

Therefore, we can write the following:

$$\begin{aligned}\mathbf{E}[\sup_{0\leqslant t\leqslant T} \langle \eta_t^{n,N}, \phi_k\rangle^2] &\leqslant 2\mathbf{E}[\sup_{0\leqslant t\leqslant T} M_t^{n,N}(\phi_k)^2] + 2\mathbf{E}\left[\left(\int_0^T \mathscr{A}_s^{n,N}(\phi_k)\,ds\right)^2\right]\\ &\leqslant 16\mathbf{E}[\langle \eta_T^{n,N}, \phi_k\rangle^2] + 18T\int_0^T \mathbf{E}[\mathscr{A}_s^{n,N}(\phi_k)^2]\,ds\end{aligned}$$

We then have:

$$\begin{aligned}&\sup_n \sum_{k\geqslant 1} \mathbf{E}[\sup_{0\leqslant t\leqslant T} < \eta_t^{n,N}, \phi_k >^2]\\ &\quad\leqslant 16\sup_n \mathbf{E}[\|\eta_t^{n,N}\|_{-4}^2] + 18T\int_0^T \sup_n \mathbf{E}\left[\sum_{k\geqslant 1} \mathscr{A}_s^{n,N}(\phi_k)^2\right] ds\end{aligned}$$

The first term is definite by Proposition 4.1, the second is also finite because the integrand is bounded on $[0, T]$ according Lemma 4.1 and Proposition 4.1.

5. COMPACTNESS OF THE FLUCTUATION PROCESSES

In this section, we discuss the relative compactness of the fluctuation processes $\eta_\cdot^{n,N}$. We shall consider them as $H^{-4}(\mathbf{R}^6)$-valued processes and look at them on a fixed compact interval $[0, T]$. We then establish the following theorem:

Theorem 5.1. Under our intial assumptions, the laws $\{\mathscr{L}(\eta^{n,N}_{\cdot}), n \geqslant 2\}$ of the fluctuation processes are relatively compact for the weak convergence on $D([0,T], H^{-4}(\mathbf{R}^6))$ and any limit law has its support in $C([0,T]; H^{-4}(\mathbf{R}^6))$.

Proof. The space $H^{-4}(\mathbf{R}^6)$ endowed with its weak topology is a Lusin space. Moreover, the space $D([0,T]; H^{-4}(\mathbf{R}^6))$(with the associated Skorohod topology) is also a Lusin space (ref. 6, Theorem 3.2.1). To show the relative compactness of the laws of the fluctuation processes $\{\eta^{n,N}_{\cdot}, n \geqslant 2\}$, it is enough to verify the following two conditions

(a) There exists a sequence $(K_m)_{m \geqslant 1}$ of weakly compact subsets of $H^{-4}(\mathbf{R}^6)$ such that,

$$\forall m \geqslant 1, \quad \forall n \geqslant 2, \quad \mathbf{P}\{\exists t \in [0,T] : \eta^{n,N}_t \notin K_m\} \leqslant 2^{-m}$$

(b) For all $\phi \in H^4(\mathbf{R}^6)$, the real processes $\{\langle \eta^{n,N}_{\cdot}, \phi \rangle, n \geqslant 2\}$ are relatively compact.

We begin with property (a). We have to let,

$$M = \sup_n \mathbf{E}[\sup_{0 \leqslant t \leqslant T} \|\eta^{n,N}_t\|^2_{-4}] < \infty$$

and to apply Chebychev's inequality to the sets:

$$K_m = \{\eta \in H^{-4}(R^6) : \|\eta\|^2_{-4} \leqslant M 2^m\}, \qquad m \geqslant 1$$

It remains to verify property (b). To establish this property we will use the following lemma, which will be proven under our initial assumptions.

Lemma 5.1. (a) For any $\phi \in H^4(\mathbf{R}^6)$

$$\lim_{M \to \infty} \sup_n \mathbf{P}\{\sup_{0 \leqslant t \leqslant T} |\langle \eta^{n,N}_t, \phi \rangle| > M\} = 0 \tag{20}$$

(b) For any $\varepsilon > 0$ and $\phi \in H^4(R^6)$, there exists $\delta > 0$ and any integer $N_0 \geqslant 2$ such that,

$$\sup_{n \geqslant N_0} \mathbf{P}\{\sup_{s,t \in [0,T], |t-s| < \delta} |\langle \eta^{n,N}_t, \phi \rangle - \langle \eta^{n,N}_s, \phi \rangle| \geqslant \varepsilon\} \leqslant \varepsilon \tag{21}$$

Proof. Property (20) is an easy consequence of Proposition 4.2. In order to verify (21), we first introduce some notation and other tools which we will use in the sequel. For each function $f \in D([0,T];\mathbf{R})$ and $\delta > 0$ we set, as in Billingsley,[7]

$$W''(f,\delta) = \sup\{|f(t)-f(r)| \wedge |f(r)-f(s)|;\, 0 \leqslant s \leqslant t \leqslant T,\, t-s<\delta\}$$

we then have:[8]

$$\sup_{s,\,t\in[0,\,t],\,|t-s|<\delta} |f(t)-f(s)| \leqslant 2W''(f,\delta) + \sup_{0\leqslant t\leqslant T} |f(t)-f(t^-)| \tag{22}$$

We also denote

$$\tau_R^{n,N} \equiv = \tau_R^{n,N}(\phi) = \inf\{t \geqslant 0 : |\mathscr{A}_t^{n,N}(\phi)| > R\} \quad \text{and} \quad Y_t^{n,N} = \langle \eta_{t\wedge\tau_R^{n,N}}, \phi\rangle$$

Hence,

$$\lim_{R\to\infty} \sup_n \mathbf{P}\{\tau_R^{n,N} \leqslant T\} = 0 \tag{23}$$

since for $\phi \in H^4(\mathbf{R}^6)$

$$\sup_n E[\sup_{0\leqslant t\leqslant T} |\mathscr{A}_t^{n,N}(\phi)|] < \infty \tag{24}$$

Indeed, the proof of Lemma 3.1, we noted that,

$$|\langle \eta_t^{n,N} \otimes \lambda_t^N, \Lambda_1\phi\rangle| \leqslant C\, \|\eta_t^{n,N}\|_{-4}\, \|\phi\|_4 \tag{25}$$

$$|\langle \eta_t^{n,N} \otimes \eta_t^{n,N}, \Lambda_1\phi\rangle| \leqslant C\sqrt{n}\, \|\eta_t^{n,N}\|_{-4}\, \|\phi\|_4 \tag{26}$$

and

$$|\langle \eta_t^{n,N}, \Gamma_1^N\phi\rangle| \leqslant C\, \|\eta_t^{n,N}\|_{-4}\, \|\phi\|_4 \tag{27}$$

Since $|\mathscr{A}_t^{n,N}(\phi)|$ is bounded by

$$|\langle \eta_t^{n,N} \otimes \lambda_t^{n,N}, \Lambda_1\phi\rangle| + \frac{1}{\sqrt{n}} |\langle \eta_t^{n,N} \otimes \eta_t^{n,N}, \Lambda_1\phi\rangle| + |\langle \eta_t^{n,N}, \Gamma_1^N\phi\rangle| \tag{28}$$

the result Eq. (24) follows from Proposition 4.2. Furthermore, since $\tau_R^{n,N}$ is a stopping time, the processes,

$$M_t^{n,N} = Y_t^{n,N} - \int_0^{t\wedge\tau_R^{n,N}} \mathscr{A}_s^{n,N}(\phi)\, ds$$

$$S_t^{n,N} = (M_t^{n,N})^2 - \int_0^{t\wedge\tau_R^{n,N}} Q_s^{n,N}(\phi)\, ds$$

are martingales; hence it is easy to show (see ref. 9) that,

$$\sup_n \mathbf{E}[(Y_t^{n,N} - Y_r^{n,N})^2 (Y_r^{n,N} - Y_s^{n,N})^2] \leqslant \text{const.}(t-s)^2$$

for $0 \leqslant s \leqslant r \leqslant t \leqslant T$. This implies (see ref. 7, Theorem 15.6) that,

$$\lim_{\delta \downarrow 0} \sup_{n \geqslant N_0} \mathbf{P}\left\{ W''(Y_{\cdot,\delta}^{n,N}) > \frac{\varepsilon}{4} \right\} = 0 \tag{29}$$

Let us fix $\varepsilon > 0$ and $\phi \in H^4(\mathbf{R}^6)$ and afterward choose an integer N_0 large enough that when $n \geqslant N_0$,

$$\frac{4}{\sqrt{n}} \|\phi\|_\infty \leqslant \frac{\varepsilon}{2}$$

Since the probability that more than two components of $W^{n,N}(t)$ change at the same time is zero, we have

$$\mathbf{P}\left\{ \sup_{0 \leqslant t \leqslant T} |\langle \eta_t^{n,N}, \phi \rangle - \langle \eta_{t^-}^{n,N}, \phi \rangle| \leqslant \frac{4}{\sqrt{n}} \|\phi\|_\infty \right\} = 1$$

which implies that,

$$\mathbf{P}\left\{ \sup_{0 \leqslant t \leqslant T} |\langle \eta_t^{n,N}, \phi \rangle - \langle \eta_{t^-}^{n,N}, \phi \rangle| \leqslant \frac{\varepsilon}{2} \right\} = 1$$

On the other hand, because of Eq. (23) and Eq. (29) it follows that,

$$\lim_{\delta \downarrow 0} \sup_{n \geqslant N_0} \mathbf{P}\left\{ W''(\langle \eta_\cdot^{n,N}, \phi \rangle, \delta) > \frac{\varepsilon}{4} \right\} = 0$$

Hence, by inequality Eq. (22), we get:

$$\lim_{\delta \downarrow 0} \sup_{n \geqslant N_0} \mathbf{P}\{ \sup_{s,\, t \in [0, T],\, |t-s| < \delta} |\langle \eta_t^{n,N}, \phi \rangle - \langle \eta_s^{n,N}, \phi \rangle| \geqslant \varepsilon \} = 0$$

This is precisely Eq. (21), so that the proof of Theorem 5.1 is complete.

6. MARTINGALE PROBLEM

In this section, we will denote by $\mathbf{P}^{n,N}$ the law of the process $\eta_\cdot^{n,N}$ and $\mathbf{E}^{n,N}$ its expectation. If E is a topological space, its borel σ-algebra is denoted by $\mathscr{B}(E)$. We denote by N_t the canonical projection of

$D([0,T]; H^{-4}(\mathbf{R}^6))$ into $H^{-4}(\mathbf{R}^6)$ defined by: $N_t(w) = w(t)$; the σ-algebra generated by $\{N_s; 0 \leqslant s \leqslant t\}$ will be noted by $\mathscr{F}_t$. Adding to our initial assumptions the hypothesis that $\{\eta_0^{n,N}, n \geqslant 2\}$ converges to η_0^N weakly in $H^{-4}(\mathbf{R}^6)$, we prove that the sequence $\{\eta_\cdot^{n,N}, n \geqslant 2\}$ on $D([0,T]; H^{-4}(\mathbf{R}^6)))$ has a unique limit point; hence, this sequence converges. To do so, we will first show that, for each limit point, some expressions are martingales. This property will then be used to obtain the uniqueness by the means of an iteration technique.

Let $\phi \in H^4(\mathbf{R}^6)$ and $g \in C_b^3(\mathbf{R})$ and define a stochastic process, $(H_t^{g,\phi})$ as follows:

$$H_t^{g,\phi} = g(\langle N_t, \phi \rangle) - \int_0^t \{\mathscr{A}_s^N(N_s, \phi)\, g'(\langle N_s, \phi \rangle)\}\, ds$$

$$-\tfrac{1}{2} \int_0^t \{Q_t^N(\phi)\, g''(\langle N_s, \phi \rangle)\}\, ds$$

where $\mathscr{A}_s^N(N_s, \phi) = \langle N_s, \Gamma_1^N \phi \rangle + \langle N_s \otimes \lambda_s^N, \Lambda_1 \phi \rangle$ and the expression $Q_s^N(\phi)$ is defined by Eq. (15).

Note that the random variables $H_t^{g,\phi}$ form a $\mathscr{F}_t$-adapted process and that $H_t^{g,\phi}$ is integrable with respect to $\mathbf{P}^N$, where P^N is a limit point of $P^{n,N}$. The latter can be easily proven using the following proposition.

Proposition 6.1. Under our initial assumptions, we have that,

$$\sup_{0 \leqslant t \leqslant T} E^N[\|N_t\|_{-4}^2] < \infty$$

(Here $\mathbf{E}^N$ denotes the integration with respect to $\mathbf{P}^N$.)

Proof of Proposition 6.1. Since $\mathbf{P}^N$ is a limit point of $\mathbf{P}^{n,N}$, we have:

$$\mathbf{E}^N[\|N_t\|_{-4}^2] \leqslant \sup_n \mathbf{E}^{n,N}[\|N_t\|_{-4}^2]$$

$$\leqslant \sup_n \mathbf{E}[\sup_{0 \leqslant t \leqslant T} \|\eta_t^{n,N}\|_{-4}^2] < \infty$$

which proves Proposition 6.1.

The next step is to show that for any limit point $\mathbf{P}^N$ of $(\mathbf{P}^{n,N})$, the process $(H_t^{g,\phi})_{t \geqslant 0}$ is a martingale. The idea is to use some martingales for $\mathbf{P}^{n,N}$ and propagate the martingale property along a subsequence converging weakly toward $\mathbf{P}^N$. However, the process $(H_t^{g,\phi})_{t \geqslant 0}$ is not a martingale for $\mathbf{P}^{n,N}$. It is then necessary to compare it with a martingale for $\mathbf{P}^{n,N}$ and show that the difference goes to zero at infinity. Let $\phi \in H^4(\mathbf{R}^6)$ and $g \in C_b^3(\mathbf{R})$. Since $(\eta_t^{n,N})_{t \geqslant 0}$ is Markovian, the following expression,

$$g(\langle \eta_t^{n,N}, \phi \rangle) - \int_0^t C_s^{n,N}(g, \phi; \eta_s^{n,N})\, ds$$

is a martingale for $\mathbf{P}^{n,N}$, where $C_t^{n,N}$ is defined by Eq. (13). We set

$$\begin{aligned} C_s^N(g, \phi; \eta_s^{n,N}) = {} & \{\langle \eta_s^{n,N}, \Gamma_1^N \phi \rangle + \langle \eta_s^{n,N} \otimes \lambda_s^N, \Lambda_1 \phi \rangle\}\, g'(\langle \eta_s^{n,N}, \phi \rangle) \\ & + \tfrac{1}{2} \{\langle \lambda_s^N, \Gamma_2^N \phi \rangle + \langle \lambda_s^N \otimes \lambda_s^N, \Lambda_2 \phi \rangle]\}\, g''(\langle \eta_s^{n,N}, \phi \rangle) \end{aligned}$$

and observe that:

$$\begin{aligned} & C_s^{n,N}(g, \phi; \eta_s^{n,N}) - C_s^N(g, \phi; \eta_s^{n,N}) \\ & \quad = \frac{1}{\sqrt{n}} \{\langle \eta_s^{n,N} \otimes \eta_s^{n,N}, \Lambda_1 \phi \rangle\}\, g'(\langle \eta_s^{n,N}, \phi \rangle) \\ & \quad + \frac{1}{2} \{[\langle \mu_s^{n,N}, \Gamma_2^N \phi \rangle - \langle \lambda_s^N, \Gamma_2^N \phi \rangle] \\ & \quad + [\langle \mu_s^{n,N} \otimes \mu_s^{n,N}, \Lambda_2 \phi \rangle - \langle \lambda_s^N \otimes \lambda_s^N, \Lambda_2 \phi \rangle]\}\, g'(\langle \eta_s^{n,N}, \phi \rangle) + \frac{1}{12\sqrt{n}}\, a_n(s) \end{aligned}$$

where,

$$|a_n(s)| \leqslant \|g''\|_\infty\, (\langle \mu_s^{n,N}, |\Gamma_3^N \phi| \rangle + \langle \mu_s^{n,N} \otimes \mu_s^{n,N}, |\Lambda_3 \phi| \rangle)$$

We now state the following proposition.

Proposition 6.2. Under our initial assumption, for each $g \in C_b^3(R)$ and each $\phi \in H^4(\mathbf{R}^6)$,

$$\lim_n \mathbf{E}[|C_s^{n,N}(g, \phi; \eta_s^{n,N}) - C^N(g, \phi; \eta_s^{n,N})|] = 0$$

Proof. To get started, note that the integral can be easily bounded by,

$$\begin{aligned} & \frac{1}{\sqrt{n}} \|g'\|_\infty \{\mathbf{E}\, |\langle \eta_s^{n,N} \otimes \eta_s^{n,N}, \Lambda_1 \phi \rangle|\} \\ & \quad + \frac{1}{2} \|g''\|_\infty \{\mathbf{E}\, |\langle \mu_s^{n,N}, \Gamma_2^N \phi \rangle - \langle \lambda_s, \Gamma_2^N \phi \rangle| \\ & \quad + \mathbf{E}\, |\langle \mu_s^{n,N} \otimes \mu_s^{n,N}, \Lambda_2 \phi \rangle - \langle \lambda_s^N \otimes \lambda_s^N, \Lambda_2 \phi \rangle|\} + \frac{1}{12\sqrt{n}}\, \mathbf{E}\, |a_n(s)| \end{aligned}$$

We need to control each of these terms. Note that the second term goes to zero as $n \to +\infty$. This is an easy consequence of Theorem 2.2.1. An analogous reasoning also shows that the third term goes to zero at infinity. As to the last term, it can be easily shown that it goes to zero as n goes to infinity. It remains to prove that, $\mathbf{E}\,|\langle \eta_s^{n,N} \otimes \eta_s^{n,N}, \Lambda_1 \phi\rangle|$ is finite. Recall that for all $\psi \in H^4(\mathbf{R}^6) \otimes H^4(\mathbf{R}^6)$, we have:

$$|\langle \eta_s^{n,N} \otimes \eta_s^{n,N}, \psi\rangle|$$

$$\leqslant \|\eta_s^{n,N}\|_{-4}^2 \left\{ \int_{\mathbf{R}^6} \left\{ \int_{\mathbf{R}^6} |D^{\alpha}_{x,v,x',v'} \psi(x,v,x',v')|^2 \, dx'\, dv' \right\} dx\, dv \right\}$$

By Lemma B, $\Lambda_1 \phi$ belongs to $H^4(\mathbf{R}^6) \otimes H^4(\mathbf{R}^6)$, so that

$$\mathbf{E}\,|\langle \eta_s^{n,N} \otimes \eta_s^{n,N}, \Lambda_1 \phi\rangle| \leqslant C_\phi \mathbf{E}[\|\eta_s^{n,N}\|_{-4}^2]$$

$$\leqslant C_\phi \mathbf{E}[\sup_{0 \leqslant s \leqslant T} \|\eta_s^{n,N}\|_{-4}^2] < \infty$$

which ends the proof of Proposition 6.2. We can now prove the main result of this section.

Theorem 6.1. Under our initial assumptions, every limit point of $(\mathbf{P}^{n,N})$ solves the following martingale problem: for every $g \in C_b^3(\mathbf{R})$ and $\phi \in H^4(\mathbf{R}^6)$, the process $(H_t^{g,\phi})$ is a $(\mathbf{P}^{\mathrm{N}}, \mathscr{F}_{\mathrm{t}})$-martingale.

Proof. Let $(\mathbf{P}^{n',N})$ be a subsequence of $(\mathbf{P}^{n,N})$ converging weakly toward $\mathbf{P}^N$. We then need to show that for each bounded, continuous, and $\mathscr{F}_s$-measurable Ψ on $D([0,T]; H^{-4}(\mathbf{R}^6))$, $E^N[H_t^{g,\phi}\Psi] = \mathbf{E}^N[H_s^{g,\phi}\Psi]$. Write: $H_t^{g,\phi} = L_t^{n,N} + B_t^{n,N}$ where

$$L_t^{n,N} = g(\langle N_t, \phi\rangle) - \int_0^t C_s^{n,N}(g; \phi, N_s)\, ds$$

We know that $\{L_t^{n,N}, t \in [0,T]\}$ is a $(\mathbf{P}^{\mathrm{n,N}}, \mathscr{F}_{\mathrm{t}})$-martingale. We can then write:

$$\mathbf{E}^{n',N}[H_t^{g,\phi}\Psi] = \mathbf{E}^{n',N}[H_s^{g,\phi}\Psi] + \mathbf{E}^{n',N}[(B_t^{n',N} - B_s^{n',N})\,\Psi]$$

with

$$|\mathbf{E}^{n',N}[(B_t^{n',N} - B_s^{n',N})\,\Psi]| \leqslant \|\Psi\|_\infty \, \{\mathbf{E}^{n',N}[|B_t^{n',N}|] + \mathbf{E}^{n',N}[|B_s^{n',N}|]\}$$

which goes to zero as n goes to zero because of Proposition 6.2. It suffices to prove that

$$\lim_{n' \to \infty} \mathbf{E}^{n', N}[H_t^{g,\phi} \Psi] = \mathbf{E}^N[H_t^{g,\phi} \Psi], \qquad \forall t \in [0, T] \tag{30}$$

This can be easily obtained with the help of Proposition 6.1 and 6.2, the $\mathbf{P}^N$-almost sure continuity of the mapping $w \to \langle N_s(w), \phi \rangle$, and the continuous application theorem.

7. CONVERGENCE OF FLUCTUATION PROCESSES

In this section, we formulate and prove the convergence of fluctuation processes. To this end, we introduce a lemma which is concerned with functional ξ_t, $t \in [0, T]$ defined by,

$$\xi_t(\phi) = \langle N_t - N_0, \phi \rangle - \int_0^t \mathscr{A}_s(N_s, \phi)\, ds \tag{31}$$

where $N_{.} \in C([0, T]; H^{-4}(\mathbf{R}^6))$.

Lemma 7.1. Assume that our initial assumptions are satisfied and let P^N be a limit point of $(\mathbf{P}^{n,N})$. For each $\phi \in H^4(\mathbf{R}^6))$, we have:

$$\mathbf{E}^N[e^{i\xi_t(\phi)} \mid \mathscr{F}_t] = \exp\left\{ i\xi_t(\phi) - \frac{1}{2} \int_s^t Q_r^N(\phi)\, dr \right\} \tag{32}$$

where $\mathbf{E}^N[\,.\mid \mathscr{F}_t]$ is the conditional expectation corresponding to P^N.

Proof. The proof of this lemma is omitted. It follows along the same line as the one in Uchiyama, see, e.g., ref. 3, or Stroock–Varadhan, see, e.g., ref. 10.

Theorem 7.1. Assume that our initial assumptions are satisfied and that $\langle \eta_0^{n,N}, \phi \rangle$ converges in distribution for all $\phi \in H^4(\mathbf{R}^6)$. Then the sequence $(P^{n,N})$ converges weakly to a probability measure concentrated on $C([0, T]; H^{-4}(\mathbf{R}^6))$.

Proof. Let $\Omega = C([0, T]; H^{-4}(\mathbf{R}^6))$ and $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ be two limit points of $(\mathbf{P}^{n,N})$ defined on Ω. We define,

$$\mathscr{F}_t^* = \sigma\{\xi_s(\phi) \colon 0 \leqslant s \leqslant t, \phi \in H^4(\mathbf{R}^6)\} \qquad \mathscr{F}^* = \sigma\{\mathscr{F}_t^* \colon t \in [0, T]\}$$

Notice that $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ coincide on $\mathscr{F}_0$. Since the relation Eq. (32) determines $\mathbf{P}^{(j)}$ on $\mathscr{F}^*$ when conditioned on $\mathscr{F}_0$, the coincidence of $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ on $\mathscr{F}_0$ implies that on $\mathscr{F}^* \vee \mathscr{F}_0 = \sigma(\mathscr{F}^*, \mathscr{F}_0)$, $\mathbf{P}^{(1)} \mid \mathscr{F}^* \vee \mathscr{F}_0 = \mathbf{P}^{(2)} \mid \mathscr{F}^* \vee \mathscr{F}_0$. Denote this common distribution on $(\Omega, \mathscr{F}^* \vee \mathscr{F}_0)$ by $\mathbf{P}^*$. Since Ω with the topology of uniform convergence on $[0, T]$ is Lusin and $\mathscr{F}$ restricted on it coincides with its topological Borel field, there exists a regular conditional probability measure $Q_{N_\cdot}^{(i)}(.)$ of $\mathbf{P}^{(i)}(i = 1, 2)$ given $\mathscr{F}^* \vee \mathscr{F}_0$. Now we let $\bar{\Omega} = \Omega \times \Omega$ and $\tilde{\mathscr{F}} = \mathscr{F} \otimes \mathscr{F}$ and define a probability measure $\tilde{\mathbf{P}}$ on $(\tilde{\Omega}, \tilde{\mathscr{F}})$ via the relation $\tilde{\mathbf{P}}(A \times B) = \int Q_{N_\cdot}^{(1)}(A)\, Q_{N_\cdot}^{(2)}(B)\, \mathbf{P}^*(dN_\cdot)$, for $A, B \in \mathscr{F}$. Clearly marginals of $\tilde{\mathbf{P}}$ agree with $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$, ie if Φ is an $\mathscr{F}$-measurable bounded function then, $\int \Phi(N_\cdot^{(j)})\, d\tilde{\mathbf{P}} = \mathbf{E}^{(j)}\Phi$, $(j = 1, 2)$. This implies, $|\mathbf{E}_{(1)}\Phi - \mathbf{E}^{(2)}\Phi| \leqslant \int |\Phi(N_\cdot^{(1)}) - \Phi(N_\cdot^{(2)})|\, d\tilde{\mathbf{P}}$. Therefore it suffices to prove $\tilde{\mathbf{P}}\{N_\cdot^{(1)} = N_\cdot^{(2)}\} = 1$ or equivalently,

$$\tilde{\mathbf{E}}[|\langle N_\cdot^{(1)}, \phi\rangle - \langle N_\cdot^{(2)}, \phi\rangle|] = 0 \tag{33}$$

for all $t \in [0, T]$ and $\phi \in H^4(\mathbf{R}^6)$. By definition of $\tilde{\mathbf{P}}$, we have

$$\tilde{\mathbf{P}}\{\langle N_t^{(1)} - N_t^{(2)}, \phi\rangle\} = \int_0^t \{[\mathscr{A}_s^N N_s^{(1)}(\phi) - \mathscr{A}_s^N N_s^{(2)}(\phi)]\, ds\} = 1$$

which implies that,

$$\begin{aligned}
&\tilde{E}[|\langle N_\cdot^{(1)}, \phi\rangle - \langle N_\cdot^{(2)}, \phi\rangle|] \\
&\quad \leqslant \int_0^t \tilde{E}[|\langle N_s^{(1)} - N_s^{(2)}, \Gamma_1^N \phi\rangle + \langle (N_s^{(1)} - N_s^{(2)}) \otimes \lambda_s^N, \Lambda_1 \phi\rangle|] \\
&\quad = \int_0^t \tilde{E}[|\langle N_S^{(1)} - N_S^{(2)}, \Gamma_1^N \phi + \mathscr{L}(\lambda_s^N)\, \phi\rangle|]\, ds
\end{aligned}$$

Since $\Gamma_1^N \phi$ and $\mathscr{L}(\lambda_s^N)\, \phi$ are both in $H^4(\mathbf{R}^6)$, their sum denoted by $L_s^N \phi$ belongs to $H^4(\mathbf{R}^6)$ so that we can iterate to obtain:

$$\begin{aligned}
&\tilde{\mathbf{E}}[|\langle N_t^{(1)}, \phi\rangle - \langle N_t^{(2)}, \phi\rangle|] \\
&\quad \leqslant \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} \tilde{\mathbf{E}}[|\langle N_{t_m}^{(1)} - N_{t_m}^{(2)}, L_{t_m}^N \cdots L_{t_1}^N(\phi)\rangle|]\, dt_m
\end{aligned}$$

The norm of the operator L_t is bounded by C for all t; Proposition 6.1 implies that,

$$\tilde{\mathbf{E}}[|\langle N_{t_m}^{(1)} - N_{t_m}^{(2)}, L_{t_m}^N \cdots L_{t_1}^N(\phi)\rangle|] \leqslant 2C^m C_T\, \|\phi\|_4$$

which concludes, $\tilde{\mathbf{E}}[|\langle N_t^{(1)} - N_t^{(2)}, \phi\rangle|] = 0$, proving Eq. (33). Thus, Theorem 7.1 has been proved.

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